Q1 Intro. To M/M/m/m model

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It is important to know which system variables (λ, μ, H) corresponds to the provided information.

Total call request rate = 3 calls/hr = λ

Average call duration = 5 minutes = $H = \frac{1}{\mu}$

Now, recall the M/M/m/m model:

The call request process is assumed to be a Poisson process which has two main properties:

1. The number of call requests in nonoverlapping intervals are i.i.d. Poisson RV whose mean = $\lambda \times$ duration of the corresponding interval.
2. The lengths of time between adjacent call requests are i.i.d. exponential RVs whose mean = $\frac{1}{\lambda}$

$$f_W(w) = \begin{cases} \lambda e^{-\lambda w}, & w > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The call durations are i.i.d. exponential RVs whose mean = $H = \frac{1}{\mu}$.

$$f_D(d) = \begin{cases} \mu e^{-\mu d}, & d > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is useful to remember also that $\underbrace{P[D > d]}_{\text{CCDF}} = e^{-\mu d}$ for $d > 0$.

Caution: Because the exponential distribution (pdf) shows up in two places, we need to carefully read the question to see whether we are asked about

the call request process (the lengths of time btw adjacent call requests $\overset{\text{i.i.d}}{\sim} \mathcal{E}(\lambda)$)

or

the call duration (the lengths of calls $\overset{\text{i.i.d.}}{\sim} \mathcal{E}(\mu)$).

(a) This is about the call request process.
The interval's length is 2 hours (9 AM to 11 AM).
In M/M/m/m model, the number N of call requests in an interval of duration T is $\mathcal{P}(\lambda T)$. In particular,

$$P[N = k] = e^{-\lambda T} \frac{(\lambda T)^k}{k!}.$$

Here, $\lambda = 3$ calls/hour, $T = 2$ hour, and we want to find $P[N=2]$.

So, $\lambda T = 3$ calls/hour $\times$ 2 hours $= 6$ and

$$P[N=2] = e^{-6}\frac{6^2}{2!} = 18e^{-6} \approx 0.0446$$

(b) Again, this is about the call request process.

Let T_1 and T_2 be the lengths of I_1 and I_2, respectively.

Let N_1 and N_2 be the number of call requests in I_1 and I_2, respectively.

(b.i) $P[N_1=2 \text{ and } N_2=3] = P[N_1=2]\,P[N_2=3] = e^{-6}\frac{6^2}{2!} \times e^{-6}\frac{6^3}{3!}$

N_1 and N_2 are i.i.d. and hence independent

$\lambda T = 6$

$\approx (0.0446)\times(0.0892) \approx 0.0040$

(b.ii) $P[N_2=3 \mid N_1=2] = P[N_2=3] = e^{-6}\frac{6^3}{3!} = 36e^{-6} \approx 0.0892$

(c) This problem is about the call duration (except the last part).

(c.i) Let D be the duration of this call.

(c.i.1) 11 AM is 60 minutes after 10:00 AM

note that we convert the unit here to [minutes] because $H = \frac{1}{\mu}$ is given in [minutes].

The event "the call is still ongoing at 11 AM" is equivalent to the event $[D>60]$. Therefore, we need to find

$$P[D>60] = e^{-\mu\times 60} = e^{-\frac{1}{5}\times 60} = e^{-12} \approx 6.14\times 10^{-6}$$

(c.i.2) 10:05 AM is 5 minutes after 10:00 AM.

The event "the call ends before 10:05 AM" is equivalent to the event $[D<5]$. Therefore, we need to find $P[D<5]$.

Now, $P[D>5] = e^{-\mu\times 5} = e^{-\frac{1}{5}\times 5} = e^{-1} \approx 0.3679$

Therefore, $P[D<5] = P[D\le 5] = 1-e^{-1} \approx 0.6321$

D is a cont. RV

(c.i.3) Here, we know that $D>5$ minutes. We want to find the probability that $D>6$.

$$P[D>6 \mid D>5] = P[D>1] = e^{-\mu\times 1} = e^{-\frac{1}{5}} \approx 0.8187$$

by the memoryless property of expo. RV.

(c.ii) It is important here to realize that we are now back to the call request process. The time W to the next call is $\mathcal{E}(\lambda)$.

not $\cancel{\mu}$

11 AM is exactly one hour after 10 AM

Note that we use [hours] as our unit here because λ is given as the average number of call requests per hour.

$$P[W < 1] = P[W \le 1] = 1 - P[W > 1] = 1 - e^{-\lambda \times 1}$$

exponential RV is continuous

$$= 1 - e^{-3} \approx 0.9502$$

2) Complete the following M/M/m/m description with the following terms:

(I) Bernoulli	(II) binomial	(III) exponential
(IV) Gaussian	(V) geometric	(VI) Poisson

The Erlang B formula is derived under some assumptions. Two important assumptions are (1) the call request process is modeled by a/an ______(A) Poisson________ process and (2) the call durations are assumed to be i.i.d. ______(B) exponential________ random variables. For the call request process, the times between adjacent call requests can be shown to be i.i.d. ______(C) exponential________ random variables. On the other hand, if we consider non-overlapping time intervals, the numbers of call requests in these intervals are ______(D) Poisson________ random variables.

In order to analyze or simulate the system described above, we consider slotted time where the duration of each time slot is small. This technique shifts our focus from continuous-time Markov chain to discrete-time Markov chain. In the limit, for the call request process, only one of the two events can happen during any particular slot: either (1) there is one new call request or (2) there is no new call request. When the slots are small and have equal length, the numbers of new call requests in the slots can be approximated by i.i.d. ______(E) Bernoulli________ random variables. In which case, if we count the total number of call requests during n slots, we will get a/an ______(F) binomial________ random variable because it is a sum of i.i.d. ______(E) Bernoulli________ random variables.

When we consider a particular time interval I (not necessarily small), the number of slots in this interval will increase as the slots get smaller. In the limit, the number of call requests in the time interval I which we approximated by a ______(F) binomial________ random variable before will approach a/an ______(D) Poisson________ random variable.

Similarly, if we consider the numbers of slots between adjacent call requests, these number will be i.i.d. ______(G) geometric________ random variables. These random variables can be thought of as discrete counterparts of the i.i.d. ______(C) exponential________ random variables in the continuous-time model.